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Gödel metric as a squashed anti–de Sitter geometry

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Abstract

We show that the non flat factor of the Gödel metric belongs to a one parameter family of 2+1 dimensional geometries that also includes the anti–de Sitter metric. The elements of this family allow a generalization à la Kaluza–Klein of the usual 3+1 dimensional Gödel metric. Their lightcones can be viewed as deformations of the anti–de Sitter ones, involving tilting and squashing. This provides a simple geometric picture of the causal structure of these space–times, anti–de Sitter geometry appearing as the boundary between causally safe and causally pathological spaces. Furthermore, we construct a global algebraic isometric embedding of these metrics in 4+3 or 3+4 dimensional flat spaces, thereby illustrating in another way the occurrence of the closed timelike curves.

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Gödel's cosmological model [1, 2] has been among the most intriguing exact solutions to Einstein equations [3] and remains still today quite interesting both mathematically and physically. The feature that has contributed most to its fame is the occurrence of closed timelike curves through each of its points [4]. This a priori limits its physical relevance, as such spaces are usually unstable with respect to quantum fluctuations [5]. However, occurrences of causality breakdowns are intrinsically interesting to study (see for instance the huge number of papers devoted to time machines [6]). Moreover, causal pathologies are large scale deficiencies, and thus parts of these spaces surrounded by more standard space–times can represent (rotating) objects with physical meaning [7, 8].

In this paper, we focus on geometric properties of a one parameter family of 2+1 dimensional metrics connecting the non trivial part of the Gödel metric to the anti–de Sitter (AdS) geometry. More precisely, by expressing the metrics of this family in an appropriate coframe, we show that they can be interpreted as resulting from a directional squashing of the lightcones of AdS space. We give the explicit form of their Killing vector fields, whose algebra corresponds to the breaking of the $so(2, 2)$ AdS isometry group into $so(2, 1) \times so(2)$. We then build a normal geodesic coordinate system so as to clarify the geometric meaning of the space–times we consider. We finally obtain a picture of these spaces as the intersection of 4 quadratic surfaces in a flat 7–dimensional space, thereby clarifying the origin of the closed timelike curves. Note that the family of metrics we examine has been considered from a different point of vue a long time ago as homogeneous cylindrically metrics, solutions of Einstein–Maxwell [9]/ Einstein–Maxwell–scalar [10] field equations, and more recently in the framework of a low energy string effective action [11].

The 3+1 dimensional Gödel metric can be expressed as the direct riemannian sum $dz^2 + d\sigma^2$ of a flat factor and the 2+1 dimensional metric :

$$d\sigma^2 = -dt^2 + dx^2 - 2\, e^{x/a} dt dy - \frac{1}{2} e^{2x/a} dy^2 \qquad . \tag{1}$$

This four dimensional geometry is G_5 invariant, and solves Einstein equations with as sources a negative cosmological constant Λ and a pressureless perfect fluid of constant energy density (dust) :

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \qquad , \qquad T_{\mu\nu} = \rho u_\mu u_\nu \qquad , \qquad \rho = -2\Lambda \qquad . \tag{2}$$

Hereafter we shall restrict ourselves to the (non-trivial) three dimensional part of the metric, $d\sigma^2$ [eq.(1)], which we shall still call the Gödel metric. We embed it in a one parameter family of geometries. To this end, let us introduce the triad :

$$\theta^0 = dt + e^{x/a} dy \quad , \quad \theta^1 = dx \quad , \quad \theta^2 = e^{x/a} dy \quad , \tag{3}$$

and consider the set of metrics

$$\begin{aligned} d\sigma_\mu^2 &= -(\theta^0)^2 + (\theta^1)^2 + \frac{1}{\mu^2}(\theta^2)^2 \\ &= -dt^2 + dx^2 - 2\, e^{x/a} dt dy + \frac{1 - \mu^2}{\mu^2} e^{2x/a} dy^2 \qquad . \end{aligned} \tag{4}$$

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The Gödel metric corresponds to $\mu^2 = 2$. The Ricci tensors of these metrics have as non vanishing components with respect to the basis (3) :

$$R_0^0 = -\frac{\mu^2}{2a^2} \quad , \quad R_1^1 = R_2^2 = \frac{\mu^2 - 2}{2a^2} \quad . \tag{5}$$

Hence, only the space with $\mu^2 = 1$ has constant (negative) curvature. It corresponds (at least locally) to an AdS space. Note also that the Cotton-York tensors of these spaces are diagonal :

$$C_0^0 = \frac{\mu(1 - \mu^2)}{a^3} \quad , \quad C_1^1 = C_2^2 = -\frac{\mu(1 - \mu^2)}{2\,a^3} \quad , \tag{6}$$

and vanish only for $\mu^2 = 1$, indicating that, in this family, only the AdS metric is conformally flat.

A standard calculation shows that the metrics $d\sigma_\mu^2$ admit at least the four Killing vector fields :

$$\begin{aligned} \xi_1 &= \partial_y & , \quad \xi_2 &= \partial_x - \tfrac{y}{a}\partial_y \quad , \\ \xi_3 &= -\mu^2 e^{-x/a}\partial_t + \tfrac{y}{a}\partial_x + (\tfrac{\mu^2}{2}e^{-2x/a} - \tfrac{y^2}{2\,a^2})\partial_y & , \quad \xi_4 &= \partial_t \quad . \end{aligned} \tag{7}$$

These vectors obey the $so(2,1) \times so(2)$ algebra :

$$[\xi_1, \xi_2] = -\frac{1}{a}\xi_1 \quad , \qquad [\xi_1, \xi_3] = +\tfrac{1}{a}\xi_2 \quad , \qquad [\xi_2, \xi_3] = -\frac{1}{a}\xi_3 \quad ,$$
$$[\xi_1, \xi_4] = 0 \quad , \qquad [\xi_2, \xi_4] = 0 \quad , \qquad [\xi_3, \xi_4] = 0 \quad , \tag{8}$$

for which a more standard representation is obtained by using as basis the vectors ξ_4 and

$$B_1^u = \frac{a}{\sqrt{2}}(\xi_1 + \xi_3) \quad , \quad B_1^v = \frac{a}{\sqrt{2}}(\xi_1 - \xi_3) \quad , \quad L_1^z = a\xi_2 \quad . \tag{9}$$

For the AdS space, two extra Killing vectors are found :

$$\begin{aligned} \xi_5 &= \sin(t/a)\partial_t - \cos(t/a)\partial_x - e^{-x/a}\sin(t/a)\partial_y \quad , \\ \xi_6 &= \cos(t/a)\partial_t + \sin(t/a)\partial_x - e^{-x/a}\cos(t/a)\partial_y \quad . \end{aligned} \tag{10}$$

These vectors commute with ξ_1, ξ_2, ξ_3 and extend the $so(2)$ subalgebra generated by ξ_4 to the second $so(2,1)$ factor of $so(2,2)$, the well known algebra of isometries of AdS space :

$$[\xi_4, \xi_5] = +\tfrac{1}{a}\xi_6 \quad , \quad [\xi_4, \xi_6] = -\tfrac{1}{a}\xi_5 \quad , \quad [\xi_5, \xi_6] = -\tfrac{1}{a}\xi_4 \quad . \tag{11}$$

The metrics (4) can be added to $(D-3)$–dimensional Euclidean Einstein metrics $d\Sigma_\nu^2$ (metrics of constant Ricci curvature : $R_j^i = \nu\delta_j^i$) to give a D–dimensional generalization of the classical Gödel solution : $ds^2 = d\sigma_\mu^2 + d\Sigma_\nu^2$ with the values of their parameters given by :

$$a^2 = \frac{D-2}{(D-4)\rho - 4\Lambda} \quad , \quad \mu^2 = 2\frac{(D-3)\rho - 2\Lambda}{(D-4)\rho - 4\Lambda} \quad , \quad \nu = \frac{\rho + 2\Lambda}{D-2} \quad . \tag{12}$$

When $D > 4$, the $(D - 3)$ space dimensions added to the Gödel geometry are all of the same order of magnitude, and thus do not correspond to usual Kaluza-Klein cosmologies, which require 4 non compact dimensions and $D - 4$ compact ones. It is nevertheless not excluded that this kind of solutions contain some elements of physical relevance, for instance in the framework of the hot phase of the early universe, where dimensional phase transitions could occur. When $D = 4$, we have $\nu = 0$, and the flat factor can be supposed compact (a circle) or not (a line). In the latter case, we reobtain the original 4–dimensional Gödel solution. Note that, if we accept the positivity of the energy density ρ, the value of the cosmological constant is bounded from above :

$$\Lambda < \frac{D - 4}{4}\rho \qquad , \tag{13}$$

in order to ensure the positivity of a^2 and μ^2. Moreover, if we impose ν to be positive[4], the values of Λ are restricted to the interval $[-\rho/2, (D - 4)\rho/4]$.

As shown above, the metric (4) becomes for $\mu^2 = 1$:

$$d\sigma^2 = -dt^2 + dx^2 - 2e^{x/a}dtdy \tag{14}$$

and describes the geometry of an AdS space, in unusual coordinates. Our purpose now is to make the connection between these coordinates and more standard ones. We remind the reader that three dimensional AdS space can be seen as the universal covering of a hyperboloid $\mathcal{H}$ of radius $2a$:

$$\mathcal{H} \equiv -U^2 - T^2 + Y^2 + Z^2 = -4a^2 \qquad , \tag{15}$$

embedded in the four dimensional (ultrahyperbolic) flat space of metric :

$$dS^2 = -dU^2 - dT^2 + dY^2 + dZ^2 \qquad . \tag{16}$$

We shall now establish the parametrisation of this quadric leading to the expression (14) of the metric, and then generalize it to the whole one parameter family of metrics (4). To this end, we shall build an auxiliary normal geodesic coordinate system and relate it both to the $\{t, x, y\}$ coordinates and to the flat four dimensional coordinates $\{U, T, Y, Z\}$, denoted generically $\{X^\alpha\}$. This will give us the required parametrisation.

The geodesics of AdS space, seen as the hyperboloid $\mathcal{H}$, are given by the "great circles" defined by the intersection of $\mathcal{H}$ with two planes passing through its center. For our purpose, it is sufficient to restrict ourselves to timelike geodesics. Indeed, the coordinate transformation is analytic and can thus be analytically continued to all types of geodesics, once obtained for timelike ones. The parametric equations of the circle passing through the origin [the point of flat coordinates $(X^\alpha(0)) = (2a, 0, 0, 0)$] and that of coordinates $(X^\alpha_\star)$ is

$$X^\alpha[\lambda] = X^\alpha[0] \cos(\frac{\lambda}{2a}) + \dot{X}^\alpha[0] \sin(\frac{\lambda}{2a}) \qquad . \tag{17}$$

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[4]Though compact spaces can have negative curvature, they are obtained by identification of a non compact universal covering and thus look less natural.

The parameter λ is the length measured along this geodesic from the origin to the point of coordinates $(X^\alpha[\lambda])$ and $(\dot{X}^\alpha[0])$ are the components of the tangent vector of the geodesic circle at the origin, given by :

$$\dot{U}[0] = 0 \ , \ \dot{T}[0] = \frac{T_\star}{\sqrt{4a^2 - U_\star^2}} \ , \ \dot{Y}[0] = \frac{Y_\star}{\sqrt{4a^2 - U_\star^2}} \ , \ \dot{Z}[0] = \frac{Z_\star}{\sqrt{4a^2 - U_\star^2}} \ . \tag{18}$$

The condition of reality $U_\star^2 < 4a^2$ ensures that the point of coordinates $(X_\star^\alpha)$ is inside the null cone whose vertex is at the origin $(2a, 0, 0, 0)$. The distance s between this point and that of coordinates $(X_\star^\alpha)$ is given by :

$$\cos(\frac{s}{2\,a}) = 2\,a\,U_\star \ . \tag{19}$$

In terms of $\{t, x, y\}$ coordinates, the same geodesic curves read :

$$\lambda + s_0 \ = \ a\arcsin\left(\frac{e^{(x[\lambda]/a)} - C_1\,C_2}{C_1\sqrt{C_2^2 - 1}}\right) \ , \tag{20}$$

$$t[\lambda] + t_0 \ = \ a\arcsin\left(\frac{C_2 - C_1 e^{-(x[\lambda]/a)}}{\sqrt{C_2^2 - 1}}\right) \ , \tag{21}$$

$$y[\lambda] + y_0 \ = \ -\frac{a}{C_1}\sqrt{C_2^2 - 1}\cos\left(\frac{t[\lambda] + t_0}{a}\right) \ , \tag{22}$$

where the constants of motion C_1 and C_2 are the first integrals obtained from the Killing vector fields ξ_1 and ξ_2. The other integration constants s_0, t_0 and y_0 are chosen such that the geodesics start $(\lambda = 0)$ at the point $(t = 0, x = 0, y = 0)$. Indeed, the transitive isometry group of AdS space allows to make such a choice without loss of generality and even fix the orientation of the axis such that :

$$\dot{T}[0] = C_2 \quad , \quad \dot{Z}[0] = C_2 - C_1 \quad , \quad \dot{Y}[0] = \pm\sqrt{(C_1^2 - 1) - (C_2 - C_1)^2} \ . \tag{23}$$

From equations (18,20–23) we obtain, after some algebra, the link between the flat and $\{t, x, y\}$ coordinate systems :

$$\exp(x/a) \ = \ \frac{(U + Y)^2 + (T - Z)^2}{4a^2} \ , $$

$$y \ = \ 2\,a\frac{YT + UZ}{(U + Y)^2 + (T - Z)^2} \ , $$

$$\sin(\frac{t + t_0}{a}) \ = \ 2\frac{T(Y^2 + YU + Z^2 - ZT) + 2\,a^2 Z}{[(U + Y)^2 + (T - Z)^2]\sqrt{Y^2 + Z^2}} \ . \tag{24}$$

The initial values t_0 and y_0 are undefined in terms of the X^α coordinates, but if we use the standard parametrisation of AdS space defined by :

$$U = 2a\cosh(r)\cos(\tau) \quad , \quad T = 2a\cosh(r)\sin(\tau) \quad , $$

$$Y = 2a\sinh(r)\cos(\theta) \quad , \quad Z = 2a\sinh(r)\sin(\theta) \quad , \tag{25}$$

we obtain

$$y_0 = -a \frac{Y_\star}{T_\star - Z_\star} \quad , \quad t_0 = a\,\theta_\star \quad , \tag{26}$$

and the coordinate transformation (24) can be rewritten :

$$\exp(x/a) = \cosh(2r) + \cos(\theta + \tau)\sinh(2r) \quad , \tag{27}$$
$$y\exp(x/a) = a\sinh(2r)\sin(\theta + \tau) \quad , \tag{28}$$
$$\sin(\frac{t}{a} + \theta) = \frac{\cosh(2r)\cos(\theta + \tau)\sin(\tau) + \cos(\tau)\sin(\theta + \tau) + \sinh(2r)\sin(\tau)}{\cosh(2r) + \cos(\theta + \tau)\sinh(2r)} \tag{29}$$

The last equation may be expressed in the more simple, but equivalent, form :

$$\tan[\frac{1}{2}(\frac{t}{a} + \theta - \tau)] = \exp(-2r)\tan[\frac{1}{2}(\theta + \tau)] \quad . \tag{30}$$

In eqs (20–22) the natural range of the variables λ and t is the interval $[0, 2a\pi]$, whereas in eq. (17) the range of λ is obviously $[0, 4a\pi]$. This reflects the fact that the metric (14), with t restricted to $[0, 2a\pi]$ actually describes a projective AdS space. This is also clear from eqs. (27–29) which show that the coordinates $\{t, x, y\}$ are left unchanged with respect to the transformation $(\theta, \tau) \mapsto (\theta + \pi, \tau + \pi)$, i.e. $X_\star^\alpha \mapsto -X_\star^\alpha$. However, if we let t run over the whole real line, we obtain the usual (universal covering) AdS space.

In terms of the $\{\tau, \theta, r\}$ coordinates, the AdS metric takes the well known form :

$$d\sigma^2 = 4a^2[-\cosh^2(r)d\tau^2 + dr^2 + \sinh^2(r)d\theta^2] \quad , \tag{31}$$

adapted to the Killing vectors :

$$\partial_\tau = a(\frac{1}{2}\xi_1 - \xi_3 + \xi_4) \quad , \quad \partial_\theta = a(\frac{1}{2}\xi_1 - \xi_3 - \xi_4) \quad . \tag{32}$$

This suggests to perform a similar transformation for arbitrary values of the μ parameter, but the resulting metric presents a conical singularity. For arbitrary values of μ, the coordinate transformation (27-30) generalizes as follows [10] :

$$\exp(x/a) = \cosh(2r) + \cos(\frac{\theta + \tau}{\mu})\sinh(2r) \quad , \tag{33}$$

$$y\exp(x/a) = \mu a\sinh(2r)\sin(\frac{\theta + \tau}{\mu}) \quad , \tag{34}$$

$$\tan[\frac{1}{2}(\frac{t}{\mu a} + \frac{\theta - \tau}{\mu})] = \exp(-2r)\tan[\frac{1}{2}(\frac{\theta + \tau}{\mu})] \quad , \tag{35}$$

and leads to the metrics :

$$d\sigma_\mu^2 = 4a^2[-\cosh^2(r)d\tau^2 + dr^2 + \sinh^2(r)d\theta^2 + \frac{1 - \mu^2}{\mu^2}\sinh^2(r)\cosh^2(r)(d\tau + d\theta)^2] \quad . \tag{36}$$

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The τ and θ coordinates are now adapted to the Killing vectors :

$$a(\frac{1}{2}\xi_1 - \frac{1}{\mu^2}\xi_3 + \xi_4) = \partial_\tau \quad , \quad a(\frac{1}{2}\xi_1 - \frac{1}{\mu^2}\xi_3 - \xi_4) = \partial_\theta \quad . \tag{37}$$

Introducing the new angular variable $\phi = (\theta + \tau)/\mu$, we finally obtain :

$$d\sigma_\mu^2 = 4a^2 \left[-d\tau^2 + dr^2 + [\sinh^2(r) + (1 - \mu^2)\sinh^4(r)]d\phi^2 - 2\mu\sinh^2(r)d\tau d\phi \right] \quad , \tag{38}$$

which, for $\mu^2 = 2$, yields the coordinate transformation and metric obtained fifty years ago by Gödel [1]. Note that the angle ϕ has to vary between 0 and 2π, otherwise the coordinate transformation (33-35) is no longer one-to-one, and the metrics present conical singularities. This explains the presence of the parameter μ in the argument of the trigonometric functions used in eqs (33-35) and as prefactor of the Killing vector ξ_3 in eqs (37). As a consequence, in terms of $\{\tau, r, \theta\}$ coordinates, we have to identify points whose θ values differ by $2\pi\mu$.

The Gödel metric is known to contain closed timelike curves, whereas AdS space (at least its universal covering) not. Examination of eq. (38) shows that the closed ϕ curves become timelike for values of $r > r_c(\mu) = \frac{1}{2}\log\frac{\mu+1}{\mu-1}$, where μ is assumed to be positive. Moreover, as the isometry group acts transitively on the space–time, such closed curves pass through all points. Hence, there exist closed timelike curves for all values of $\mu > 1$. This can be visualized geometrically by reinterpreting the metrics (4) as a family of tensor fields $\mathbf{g}_\mu$ on an AdS background space and comparing the different apertures of the lightcones $\mathcal{C}_\mu \equiv \mathbf{g}_\mu(\vec{v}, \vec{v}) = 0$ as function of μ and their location. With respect to the basis $\{\vec{e}_0, \vec{e}_1, \vec{e}_2\}$, dual of the triad (3), these cones have everywhere the same shape. They are stretched in the $\vec{e}_2$ direction for $\mu > 1$, and squashed for $\mu < 1$ (see Fig.1). On the other hand, if this frame is boosted to the basis $\{\vec{e}_\tau = \text{sech}(r)\partial_\tau, \vec{e}_r = \partial_r, \vec{e}_\theta = \text{csch}(r)\partial_\theta\}$, the cones appear deformed with respect to the (invariant) AdS lightcone (Figs. 1-3). As the r coordinate increases, the cones with $\mu < 1$ narrow down and tilt towards the $(-\vec{e}_\theta)$ direction; in the limit of $r \mapsto \infty$ they coincide with the AdS null direction $\vec{e}_\tau + \vec{e}_\theta$. The cones with $\mu > 1$, as for them, open out in the $\vec{e}_\theta$ direction; once $r > r_c(\mu)$, they include the $\vec{e}_\theta$ axis. As a consequence, the circles of radius $r > r_c(\mu)$ have their tangent vectors everywhere inside $\mathcal{C}_\mu$, thus yielding closed timelike curves.

From eq. (38) it is easy to compute the geodesics and extend the analysis already performed for AdS and Gödel's metrics [12, 4]. We find that for $\mu > 1$, the qualitative behavior of the geodesics are the same as in Gödel's space. The light geodesics starting from a point spiral up, diverging up to a caustic circle of radius $r = \frac{1}{2}\log\frac{\mu+1}{\mu-1}$ (measured on a surface of constant τ) and then reconverge. For $\mu = 1$, the closed timelike curves are pushed to infinity, but there still exist Cauchy horizons, a well known property of AdS space. For $\mu < 1$ we have not detected any causal pathology.

Another way to understand the presence of closed time lines is to consider isometric embeddings of the metrics (38) in flat spaces. Though the three dimensional AdS space can easily be embedded in four dimensional flat space [eq.(25)], the best result we have obtained for the other spaces of the family is the almost obvious, but

global, embedding in seven dimensional flat spaces of metrics :

$$dS^2 = -dU^2 - dT^2 + dY^2 + dZ^2 + \varepsilon(dA^2 + dB^2 - dC^2) \quad , \quad \varepsilon = \pm 1 = \text{sign}(1 - \mu^2) \tag{39}$$

via the parametrisation :

$$U = 2\,\mu\,a\cosh(r)\cos(\frac{\tau}{\mu}) \quad , \quad T = 2\,\mu\,a\cosh(r)\sin(\frac{\tau}{\mu}) \quad ,$$

$$Y = 2\,\mu\,a\sinh(r)\cos(\frac{\theta}{\mu}) \quad , \quad Z = 2\,\mu\,a\sinh(r)\sin(\frac{\theta}{\mu})$$

$$A = a\,\sqrt{|1-\mu^2|}\sinh(2r)\cos(\frac{\tau+\theta}{\mu}) \quad , \quad B = a\,\sqrt{|1-\mu^2|}\sinh(2r)\sin(\frac{\tau+\theta}{\mu}) \quad ,$$

$$C = a\,\sqrt{|1-\mu^2|}\cosh(2r) \quad . \tag{40}$$

This embedding can be seen as the intersection of the four quadratic surfaces :

$$\begin{aligned}
\mathcal{H}_1 &\equiv -U^2 - T^2 + Y^2 + Z^2 = -4\,a^2 \quad , \\
\mathcal{H}_2 &\equiv C^2 - A^2 - B^2 = a^2\,|1-\mu^2| \quad , \\
\mathcal{P}_1 &\equiv A - \frac{\sqrt{|1-\mu^2|}}{2\,a\,\mu^2}(U\,Y - T\,Z) = 0 \quad , \\
\mathcal{P}_2 &\equiv B - \frac{\sqrt{|1-\mu^2|}}{2\,a\,\mu^2}(U\,Z + T\,Y) = 0 \quad .
\end{aligned} \tag{41}$$

We easily see that, when $\mu^2 = 1$, we recover the standard embedding of AdS space as the hyperboloid $\mathcal{H}$ (15), here in the four dimensional subspace $A = B = C = 0$. In the other cases, we see that both cylindrical surfaces $\mathcal{H}_1$ and $\mathcal{H}_2$ possess non contractible closed curves. The first is topologically equivalent to the cartesian product of the AdS hyperboloid $\mathcal{H}$ with $\mathbb{R}^3$ and can be unwinded in the τ-direction, thereby removing the closed time like curves. The second, $\mathcal{H}_2$, cannot be unwinded without introducing a singularity in the three dimensional manifolds, which do not remain homeomorphic to $\mathbb{R}^3$ at $r = 0$. The circles generated by varying the ϕ coordinate are always spacelike when $\varepsilon = +1$, that is, for $\mu^2 < 1$; for $\varepsilon = -1$, $\mu^2 > 1$, they become timelike when, in eq. (39), the term $dA^2 + dB^2$ dominates $dY^2 + dZ^2$, i.e. once their radii are large enough.

The main interest of the previous analysis resides in the geometric picture we have obtained of the Gödel–like universes. On the one hand, we have shown how these geometries can be interpreted as resulting from the squashing of AdS lightcones. On the other hand, we have obtained a simple global embedding of these spaces, showing clearly the unavoidability of closed timelike curves for spaces with $\mu^2 > 1$, AdS space ($\mu^2 = 1$) appearing as the limiting case.

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Figure caption

Figure 1. Right hand side : The cone $\mathcal{C}_{1/2}$ (in dark gray) and the cone $\mathcal{C}_2$ (in pale gray), respectively inside and around the AdS lightcone $\mathcal{C}_1$ (in medium gray), at $r = 0$, where the frames $\{\vec{e}_0, \vec{e}_1, \vec{e}_2\}$ and $\{\vec{e}_\tau, \vec{e}_r, \vec{e}_\theta\}$ are identical. Left hand side : a plane section orthogonal to $\vec{e}_\tau$ of the three nested cones.

Figure 2. Right hand side : The cone $\mathcal{C}_{1/2}$ (in dark gray) and the cone $\mathcal{C}_2$ (in pale gray), respectively inside and around the AdS lightcone $\mathcal{C}_1$ (in medium gray), at $r = 0.5 < r_c(2) \approx 0.549$. Left hand side : a plane section orthogonal to $\vec{e}_\tau$ of these three nested cones.

Figure 3. Right hand side : The cone $\mathcal{C}_{1/2}$ (in dark gray) and the cone $\mathcal{C}_2$ (in pale gray), respectively inside and around the AdS lightcone $\mathcal{C}_1$ (in medium gray), at $r = 0.7 > r_c(2) \approx 0.549$. Here only the parts above the plane $[\vec{e}_r, \vec{e}_\theta]$ have been drawn, showing that the $\vec{e}_\theta$ direction is now inside $\mathcal{C}_2$. Left hand side : a plane section orthogonal to $\vec{e}_\tau$ of these three nested cones, on which the branch of hyperbola representing the intersection of the plane with the lower half cone $\mathcal{C}_2$ has been suppressed.
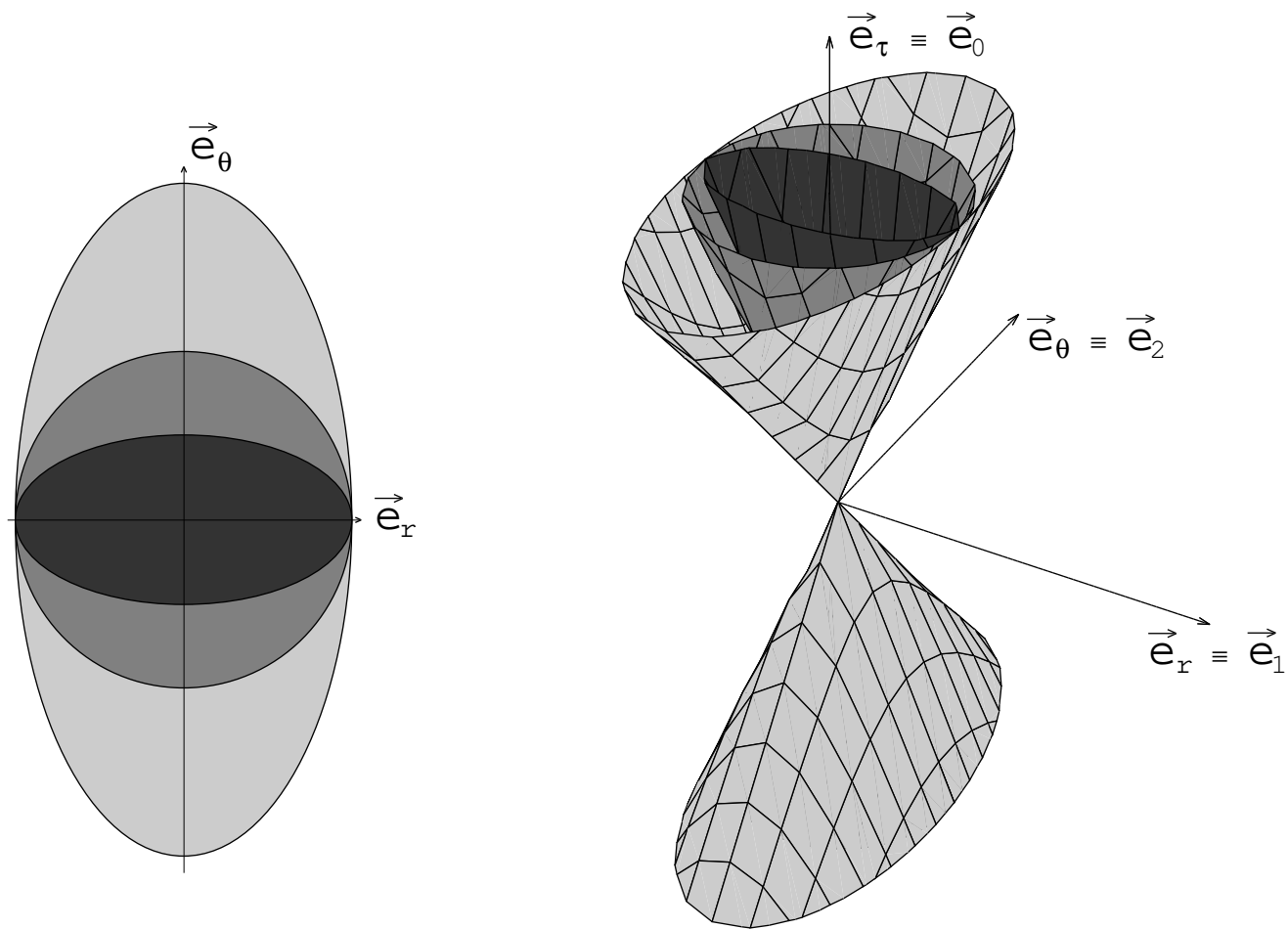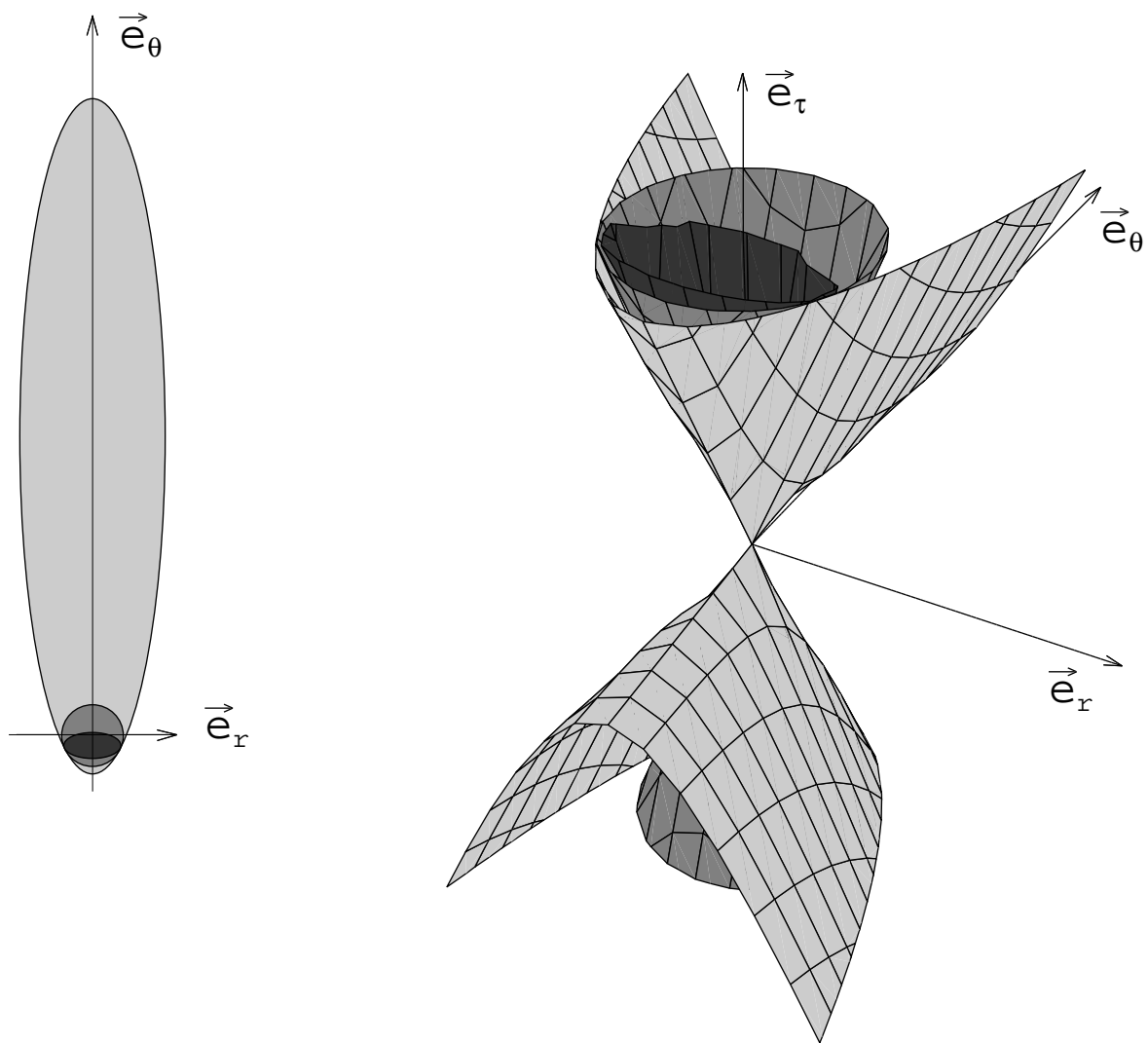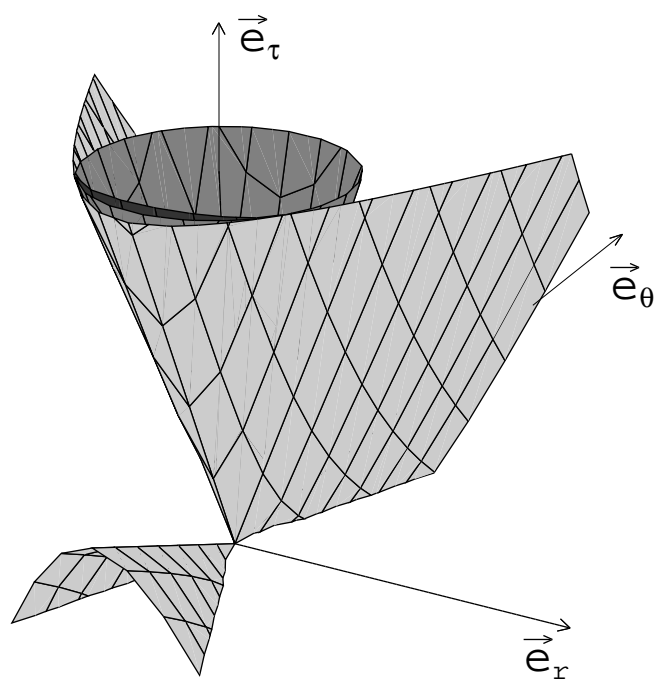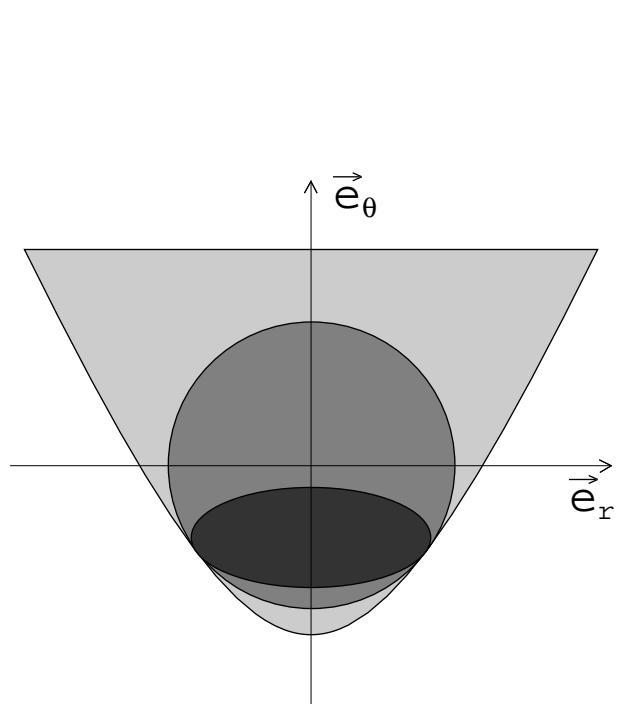
$\vec{e}_\theta$

$\vec{e}_r$

$\vec{e}_\tau \equiv \vec{e}_0$

$\vec{e}_\theta \equiv \vec{e}_2$

$\vec{e}_r \equiv \vec{e}_1$

Fig. 1

Fig. 2

Fig. 3